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This is the Published version of the following publication

Tseng, Kuei-Lin, Hwang, Shiow Ru and Dragomir, Sever S (2005) On Some New Inequalities of Hermite-Hadamard-Féjer Type Involving Convex Functions. Research report collection, 8 (4).

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ON SOME NEW INEQUALITIES OF HERMITE-HADAMARD-FEJÉR TYPE INVOLVING CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some inequalities of Hermite-Hadamard-Fejér type for m -convex functions and s -convex functions.

1. INTRODUCTION

If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known as Hermite-Hadamard inequality.

Fejér [14] gave a generalization of the inequalities (1.1) as the following:

If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, and $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, then

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx.$$

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [1] – [12], [14] – [16], [19] – [23].

Definition 1 (see [6, 13, 18]). *A function $f : [0, b] \rightarrow \mathbb{R}$ is said to be m -convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:*

$$(1.3) \quad f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

We denote the set of all m -convex functions on $[0, b]$ by $K_m(b)$.

Dragomir and Toader [13] (see also [6]) proved the following two theorems:

Theorem 1. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an m -convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[0, b]$, then*

$$(1.4) \quad \int_a^b f(x)dx \leq (b-a) \min \left\{ \frac{f(a) + mf(\frac{b}{m})}{2}, \frac{f(b) + mf(\frac{a}{m})}{2} \right\}.$$

1991 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Hermite-Hadamard inequality, m -convex functions, s -convex functions in the first and second senses.

Theorem 2. Let f, m, a and b be defined as in Theorem 1. If f is also differentiable on $(0, \infty)$, then

$$(1.5) \quad \left[\frac{f(mb)}{m} - \frac{b-a}{2} f'(mb) \right] (b-a) \leq \int_a^b f(x) dx \leq \frac{1}{2} [(b-ma)f(b) - (a-mb)f(a)].$$

The following two theorems are due to Dragomir [6]:

Theorem 3. Let f be defined as in Theorem 1. Then

$$(1.6) \quad f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx \leq \frac{b-a}{8} \left[f(a) + f(b) + 2m \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) + m^2 \left(f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right) \right) \right].$$

Theorem 4. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an m -convex function with $m \in (0, 1]$. If $f \in L_1[am, b]$ where $0 \leq a < b$, then

$$\frac{1}{m+1} \left[\int_a^{mb} f(x) dx + \frac{mb-a}{b-ma} \int_{ma}^b f(x) dx \right] \leq (mb-a) \frac{f(a) + f(b)}{2}.$$

Remark 1. A misprint of (1.6) in the original paper has been corrected here.

Definition 2 (see [9, 10, 17]). Let $0 < s \leq 1$. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the first sense, if for every $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, we have:

$$(1.7) \quad f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y).$$

We denote the set of all s -convex functions in the first sense by K_s^1 .

Definition 3 (see [9, 10, 17]). Let $0 < s \leq 1$. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense, if for every $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have the inequality (1.7). The set of all s -convex functions in the second sense is denoted by K_s^2 .

Dragomir and Fitzpatrick [9, 10] proved the following two theorems:

Theorem 5. Let $f \in K_s^1$ and $a, b \in [0, \infty)$ with $a < b$. Then

$$(1.8) \quad (b-a)f\left[2^{-\frac{1}{s}}(a+b)\right] \leq \int_a^b f(x) dx$$

and

$$(1.9) \quad f\left(\frac{a+b}{2^{\frac{1}{s}-1}}\right) \leq \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}} \left[t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right]\right) dt \leq \int_0^1 f\left(at^{\frac{1}{s}} + b(1-t)^{\frac{1}{s}}\right) dt \leq \frac{f(a) + f(b)}{2}.$$

Theorem 6. Let $f \in K_s^2$ and $a, b \in [0, \infty)$ with $a < b$. Then

$$(1.10) \quad 2^{s-1}(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x) dx \leq \frac{(b-a)(f(a) + f(b))}{(s+1)}.$$

In this paper, we shall establish some generalizations of Theorems 1 – 6.

2. MAIN RESULTS

Throughout this section, let $g : [a, b] \rightarrow \mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{a+b}{2}$.

Theorem 7. *Let f, m, a and b be defined as in Theorem 1. Then*

$$(2.1) \quad \int_a^b f(x)g(x)dx \leq \min \left\{ \frac{f(a) + mf(\frac{b}{m})}{2}, \frac{f(b) + mf(\frac{a}{m})}{2} \right\} \int_a^b g(x)dx.$$

Proof. Since f is m -convex and g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} \int_a^b f(x)g(x)dx &= \frac{1}{2} \left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx \right] \\ &= \frac{1}{2} \int_a^b [f(x) + f(a+b-x)]g(x)dx \\ &= \frac{1}{2} \int_a^b \left[f\left(\frac{b-x}{b-a}a + m\frac{x-a}{b-a} \cdot \frac{b}{m}\right) \right. \\ &\quad \left. + f\left(\frac{x-a}{b-a}a + m\frac{b-x}{b-a} \cdot \frac{b}{m}\right) \right] g(x)dx \\ &\leq \frac{1}{2} \int_a^b \left[\frac{b-x}{b-a}f(a) + m\frac{x-a}{b-a}f\left(\frac{b}{m}\right) \right. \\ &\quad \left. + \frac{x-a}{b-a}f(a) + m\frac{b-x}{b-a}f\left(\frac{b}{m}\right) \right] g(x)dx \\ &= \frac{f(a) + mf(\frac{b}{m})}{2} \int_a^b g(x)dx. \end{aligned} \tag{2.2}$$

On the other hand,

$$\begin{aligned} \int_a^b f(x)g(x)dx &= \frac{1}{2} \int_a^b [f(x) + f(a+b-x)]g(x)dx \\ &= \frac{1}{2} \int_a^b \left[f\left(m\frac{b-x}{b-a} \cdot \frac{a}{m} + \frac{x-a}{b-a}b\right) \right. \\ &\quad \left. + f\left(m\frac{x-a}{b-a} \cdot \frac{a}{m} + \frac{b-x}{b-a}b\right) \right] g(x)dx \\ &\leq \frac{1}{2} \int_a^b \left[m\frac{b-x}{b-a}f\left(\frac{a}{m}\right) + \frac{x-a}{b-a}f(b) \right. \\ &\quad \left. + m\frac{x-a}{b-a}f\left(\frac{a}{m}\right) + \frac{b-x}{b-a}f(b) \right] g(x)dx \\ &= \frac{mf(\frac{a}{m}) + f(b)}{2} \int_a^b g(x)dx. \end{aligned} \tag{2.3}$$

The inequality (2.1) follows immediately from (2.2) and (2.3). ■

Remark 2. *If we choose $g(x) \equiv 1$, then Theorem 7 reduces to Theorem 1.*

Remark 3. If $m = 1$, then the inequality (2.1) reduces to the second inequality of (1.2) where $0 \leq a < b < \infty$.

In order to prove our second theorem, we need the following lemma:

Lemma 1 (see [6] or [13]). *If f is differentiable on $[0, b]$, then $f \in K_m(b)$ if and only if*

$$(2.4) \quad f(x) - mf(y) \leq f'(x)(x - my)$$

for $x, y \in [0, b]$.

Theorem 8. *Let f, m, a and b be defined as in Theorem 2. Then*

$$(2.5) \quad \left[\frac{f(mb)}{m} - \frac{b-a}{2} f'(mb) \right] \int_a^b g(x) dx \leq \int_a^b f(x) g(x) dx \\ \leq \int_a^b [(x - ma) f'(x) + mf(a)] g(x) dx.$$

Proof. By Lemma 1, for $x \in [a, b]$, we have

$$f(mb) - mf(x) \leq f'(mb)(mb - mx)$$

and

$$f(mb) - mf(a + b - x) \leq f'(mb)[mb - m(a + b - x)],$$

so that

$$(2.6) \quad \frac{f(mb)}{m} - (b - x) f'(mb) \leq f(x)$$

and

$$(2.7) \quad \frac{f(mb)}{m} - (x - a) f'(mb) \leq f(a + b - x).$$

If we add the inequalities (2.6) and (2.7), then

$$(2.8) \quad \frac{2f(mb)}{m} - (b - a) f'(mb) \leq f(x) + f(a + b - x)$$

for all $x \in [a, b]$. Since g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, multiplying (2.8) by $\frac{g(x)}{2}$, and integrating the resulting inequalities on $[a, b]$ yields

$$\left[\frac{f(mb)}{m} - \frac{b-a}{2} f'(mb) \right] \int_a^b g(x) dx \\ \leq \frac{1}{2} \int_a^b [f(x)g(x) + f(a + b - x)g(x)] dx \\ = \frac{1}{2} \left[\int_a^b f(x)g(x) dx + \int_a^b f(a + b - x)g(a + b - x) dx \right] \\ = \int_a^b f(x)g(x) dx.$$

This proves the first inequality in (2.5). Putting in (2.4) $y = a$, we have for $x \geq ma$

$$(2.9) \quad (x - ma) f'(x) + mf(a) \geq f(x).$$

Multiplying (2.9) by $g(x)$ and integrating over x on $[a, b]$, we obtain the second inequality in (2.5). This completes the proof. ■

Remark 4. If we choose $g(x) \equiv 1$, then Theorem 8 reduces to Theorem 2.

Theorem 9. Let f, m, a and b be defined as in Theorem 3. Then

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx &\leq \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} g(x)dx \\
 &\leq \frac{1}{8} \left[f(a) + f(b) + 2m \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) \right. \\
 &\quad \left. + m^2 \left(f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right) \right) \right] \int_a^b g(x)dx \\
 (2.10) \quad &\leq \frac{m^2 [f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right)]}{2} \int_a^b g(x)dx.
 \end{aligned}$$

Proof. Since f is m -convex, $f \in L_1[a, b]$ and g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned}
 &f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \\
 &= \int_a^b f\left[\frac{1}{2}(a+b-x) + \frac{m}{2} \cdot \frac{x}{m}\right] g(x)dx \\
 &\leq \int_a^b \left[\frac{1}{2}f(a+b-x) + \frac{m}{2}f\left(\frac{x}{m}\right)\right] g(x)dx \\
 &= \int_a^b \frac{1}{2} \left[f(a+b-x)g(a+b-x) + mf\left(\frac{x}{m}\right)g(x) \right] dx \\
 &= \frac{1}{2} \left[\int_a^b f(x)g(x)dx + \int_a^b mf\left(\frac{x}{m}\right)g(x)dx \right] \\
 (2.11) \quad &= \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} g(x)dx, \\
 &= \frac{1}{4} \left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx \right. \\
 &\quad \left. + \int_a^b mf\left(\frac{x}{m}\right)g(x)dx + \int_a^b mf\left(\frac{a+b-x}{m}\right)g(a+b-x)dx \right] \\
 &= \frac{1}{8} \left[2 \int_a^b f(x)g(x)dx + 2 \int_a^b f(a+b-x)g(x)dx \right. \\
 &\quad \left. + 2 \int_a^b mf\left(\frac{x}{m}\right)g(x)dx + 2 \int_a^b mf\left(\frac{a+b-x}{m}\right)g(x)dx \right] \\
 &= \frac{1}{8} \left[\int_a^b f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}m\frac{b}{m}\right)g(x)dx + \int_a^b f\left(\frac{b-x}{b-a}m\frac{a}{m} + \frac{x-a}{b-a}b\right)g(x)dx \right. \\
 &\quad \left. + \int_a^b f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}m\frac{b}{m}\right)g(x)dx + \int_a^b f\left(\frac{x-a}{b-a}m\frac{a}{m} + \frac{b-x}{b-a}b\right)g(x)dx \right]
 \end{aligned}$$

$$\begin{aligned}
& + \int_a^b m f \left(\frac{b-x}{b-a} \frac{a}{m} + \frac{x-a}{b-a} m \frac{b}{m^2} \right) g(x) dx \\
& + \int_a^b m f \left(\frac{b-x}{b-a} m \frac{a}{m^2} + \frac{x-a}{b-a} \frac{b}{m} \right) g(x) dx \\
& + \int_a^b m f \left(\frac{x-a}{b-a} \cdot \frac{a}{m} + \frac{b-x}{b-a} \cdot m \cdot \frac{b}{m^2} \right) g(x) dx \\
& + \int_a^b m f \left(\frac{x-a}{b-a} m \frac{a}{m^2} + \frac{b-x}{b-a} \cdot \frac{b}{m} \right) g(x) dx \Big] \\
& \leq \frac{1}{8} \left\{ \int_a^b \left[\frac{b-x}{b-a} f(a) + m \frac{x-a}{b-a} f \left(\frac{b}{m} \right) \right] g(x) \right. \\
& \quad + \int_a^b \left[m \frac{b-x}{b-a} f \left(\frac{a}{m} \right) + \frac{x-a}{b-a} f(b) \right] g(x) dx \\
& \quad + \int_a^b \left[\frac{x-a}{b-a} f(a) + m \frac{b-x}{b-a} f \left(\frac{b}{m} \right) \right] g(x) dx \\
& \quad + \int_a^b \left[m \frac{x-a}{b-a} f \left(\frac{a}{m} \right) + \frac{b-x}{b-a} f(b) \right] g(x) dx \\
& \quad + \int_a^b m \left[\frac{b-x}{b-a} f \left(\frac{a}{m} \right) + m \frac{x-a}{b-a} f \left(\frac{b}{m^2} \right) \right] g(x) dx \\
& \quad + \int_a^b m \left[m \frac{b-x}{b-a} f \left(\frac{a}{m^2} \right) + \frac{x-a}{b-a} f \left(\frac{b}{m} \right) \right] g(x) dx \\
& \quad + \int_a^b m \left[\frac{x-a}{b-a} f \left(\frac{a}{m} \right) + m \frac{b-x}{b-a} f \left(\frac{b}{m^2} \right) \right] g(x) dx \\
& \quad \left. + \int_a^b m \left[m \frac{x-a}{b-a} f \left(\frac{a}{m^2} \right) + \frac{b-x}{b-a} f \left(\frac{b}{m} \right) \right] g(x) dx \right\} \\
(2.12) \quad & = \frac{1}{8} \left[f(a) + f(b) + 2m \left(f \left(\frac{a}{m} \right) + f \left(\frac{b}{m} \right) \right) \right. \\
& \quad \left. + m^2 \left(f \left(\frac{a}{m^2} \right) + f \left(\frac{b}{m^2} \right) \right) \right] \int_a^b g(x) dx \\
& = \frac{\int_a^b g(x) dx}{8} \left\{ f \left(0 \cdot a + m \cdot \frac{a}{m} \right) + f \left(0 \cdot b + m \cdot \frac{b}{m} \right) \right. \\
& \quad + 2m \left[f \left(0 \cdot \frac{a}{m} + m \cdot \frac{a}{m^2} \right) + f \left(0 \cdot \frac{b}{m} + m \cdot \frac{b}{m^2} \right) \right] \\
& \quad \left. + m^2 \left[f \left(\frac{a}{m^2} \right) + f \left(\frac{b}{m^2} \right) \right] \right\} \\
& \leq \frac{\int_a^b g(x) dx}{8} \left\{ m f \left(\frac{a}{m} \right) + m f \left(\frac{b}{m} \right) + 3m^2 \left[f \left(\frac{a}{m^2} \right) + f \left(\frac{b}{m^2} \right) \right] \right\} \\
& = \frac{\int_a^b g(x) dx}{8} \left\{ m f \left(0 \cdot \frac{a}{m} + m \frac{a}{m^2} \right) + m f \left(0 \cdot \frac{b}{m} + m \frac{b}{m^2} \right) \right. \\
& \quad \left. + 3m^2 \left[f \left(\frac{a}{m^2} \right) + f \left(\frac{b}{m^2} \right) \right] \right\} \\
(2.13) \quad & \leq \frac{m^2 \left[f \left(\frac{a}{m^2} \right) + f \left(\frac{b}{m^2} \right) \right]}{2} \int_a^b g(x) dx.
\end{aligned}$$

The inequalities (2.10) follow from (2.11), (2.12) and (2.13). ■

Remark 5. If we choose $g(x) \equiv 1$, then Theorem 9 reduces to Theorem 3.

Remark 6. If $m = 1$, then the inequalities (2.10) reduce to the inequalities (1.2) when $0 \leq a < b < \infty$.

Theorem 10. Suppose that $f : [0, \infty) \rightarrow \mathbb{R}$ is an m -convex function with $f \in L_1[ma, b]$ and $k : [ma, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{ma+b}{2}$ with $\int_{ma}^b k(x)dx > 0$, where $m \in [0, 1]$ and $0 \leq a < b$.

(a) If $a < mb$ and $h : [a, mb] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+mb}{2}$ with $\int_a^{mb} h(x)dx > 0$, then

$$(2.14) \quad \frac{1}{m+1} \left(\frac{\int_a^{mb} f(x)h(x)dx}{\int_a^{mb} h(x)dx} + \frac{\int_{ma}^b f(x)k(x)dx}{\int_{ma}^b k(x)dx} \right) \leq \frac{f(a) + f(b)}{2}.$$

(b) If $mb < a$ and $h : [mb, a] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+mb}{2}$ with $\int_{mb}^a h(x)dx > 0$, then the inequality (2.14) also holds.

Proof. (a) Since f is m -convex, $f \in L_1[ma, b]$, k is nonnegative, integrable, symmetric about $\frac{ma+b}{2}$ with $\int_{ma}^b k(x)dx > 0$, we have

$$\begin{aligned} \int_{ma}^b f(x)k(x)dx &= \frac{\int_{ma}^b f(x)k(x)dx + \int_{ma}^b f(ma+b-x)k(ma+b-x)dx}{2} \\ &= \frac{\int_{ma}^b [f(x) + f(ma+b-x)]k(x)dx}{2} \\ &= \frac{1}{2} \int_{ma}^b \left[f\left(\frac{b-x}{b-ma}ma + \frac{x-ma}{b-ma}b\right) \right. \\ &\quad \left. + f\left(\frac{x-ma}{b-ma}ma + \frac{b-x}{b-ma}b\right) \right] k(x)dx \\ &\leq \frac{1}{2} \int_{ma}^b \left[m\frac{b-x}{b-ma}f(a) + \frac{x-ma}{b-ma}f(b) \right. \\ &\quad \left. + m\frac{x-ma}{b-ma}f(a) + \frac{b-x}{b-ma}f(b) \right] k(x)dx \\ (2.15) \quad &= \frac{mf(a) + f(b)}{2} \int_{ma}^b k(x)dx. \end{aligned}$$

Similarly, we have

$$(2.16) \quad \int_a^{mb} f(x)h(x)dx \leq \frac{f(a) + mf(b)}{2} \int_a^{mb} h(x)dx$$

The inequality (2.14) follows immediately from (2.15) and (2.16).

The proof of part (b) is similar to that of part (a). ■

Remark 7. If we choose $h(x) \equiv 1$ and $k(x) \equiv 1$, then Theorem 10 reduces to Theorem 4.

Remark 8. If $m = 1$ and $h(x) = k(x) = g(x)$ on $[a, b]$, then the inequality (2.14) reduces to the second inequality of (1.2) when $0 \leq a < b < \infty$.

In order to prove our next theorem, we need the following lemma:

Lemma 2 ([17]). *If $0 < s < 1$ and $f \in K_s^1$, then f is nondecreasing on $[0, \infty)$.*

Theorem 11. *Let f, a and b be defined as in Theorem 5. Then*

$$(2.17) \quad f \left[2^{-\frac{1}{s}}(a+b) \right] \int_a^b g(x) dx \leq \int_a^b f(x) g(x) dx$$

and

$$(2.18) \quad \begin{aligned} & f \left(\frac{a+b}{2^{\frac{2}{s}-1}} \right) \int_a^b g(x) dx \\ & \leq \int_a^b \left\{ f \left[\left(\frac{1}{2} \cdot \frac{b-x}{b-a} \right)^{\frac{1}{s}} + \left(\frac{1}{2} \cdot \frac{x-a}{b-a} \right)^{\frac{1}{s}} \right] (a+b) \right\} g(x) dx \\ & \leq \int_a^b f \left[a \left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} + b \left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} \right] g(x) dx \\ & \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx. \end{aligned}$$

Proof. Since $f \in K_s^1$ and g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} f \left[2^{-\frac{1}{s}}(a+b) \right] \int_a^b g(x) dx &= \int_a^b f \left[2^{-\frac{1}{s}}x + 2^{-\frac{1}{s}}(a+b-x) \right] g(x) dx \\ &\leq \int_a^b \left[\frac{1}{2}f(x) + \frac{1}{2}f(a+b-x) \right] g(x) dx \\ &= \frac{1}{2} \left[\int_a^b f(x)g(x) dx + \int_a^b f(a+b-x)g(x) dx \right] \\ &= \frac{1}{2} \left[\int_a^b f(x)g(x) dx + \int_a^b f(a+b-x)g(a+b-x) dx \right] \\ &= \int_a^b f(x)g(x) dx. \end{aligned}$$

This proves (2.17).

Next, if $s = 1$ then (2.18) is (1.2). Let $0 < s < 1$, and $\alpha, \beta \geq 0$, then

$$\left(\frac{\alpha + \beta}{2} \right)^{\frac{1}{s}} \leq \frac{1}{2} \left(\alpha^{\frac{1}{s}} + \beta^{\frac{1}{s}} \right).$$

Now, by Lemma 2, f is nondecreasing on $[0, \infty)$. Since g is nonnegative integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} & f \left(\frac{a+b}{2^{\frac{2}{s}-1}} \right) \int_a^b g(x) dx \\ &= \int_a^b f \left[\left(\frac{1}{2} \cdot \frac{b-x}{b-a} + \frac{1}{2} \cdot \frac{x-a}{b-a} \right)^{\frac{1}{s}} 2(a+b) \right] g(x) dx \\ &\leq \int_a^b f \left[\left(\frac{1}{2} \left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} + \frac{1}{2} \left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} \right) 2(b+a) \right] g(x) dx \end{aligned}$$

$$\begin{aligned}
(2.19) \quad &= \int_a^b f \left[\left(\frac{b-x}{2(b-a)} \right)^{\frac{1}{s}} + \left(\frac{x-a}{2(b-a)} \right)^{\frac{1}{s}} \right] (a+b)g(x)dx \\
&= \int_a^b f \left\{ \left(\frac{1}{2} \right)^{\frac{1}{s}} \left[\left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} b \right] \right. \\
&\quad \left. + \left(\frac{1}{2} \right)^{\frac{1}{s}} \left[\left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} b \right] \right\} g(x)dx \\
&\leq \int_a^b \left\{ \frac{1}{2} f \left[\left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} b \right] \right. \\
&\quad \left. + \frac{1}{2} f \left[\left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} b \right] \right\} g(x)dx \\
&= \frac{1}{2} \int_a^b f \left[\left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} b \right] g(x)dx \\
&\quad + \frac{1}{2} \int_a^b f \left[\left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} b \right] g(a+b-x)dx \\
(2.20) \quad &= \int_a^b f \left[\left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} b \right] g(x)dx.
\end{aligned}$$

On the other hand, using (1.7) we have

$$\begin{aligned}
&\int_a^b \left\{ \frac{1}{2} f \left[\left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} b \right] \right. \\
&\quad \left. + \frac{1}{2} f \left[\left(\frac{x-a}{b-a} \right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a} \right)^{\frac{1}{s}} b \right] \right\} g(x)dx \\
&\leq \frac{1}{2} \int_a^b \left[\frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \right] g(x)dx \\
&\quad + \frac{1}{2} \int_a^b \left[\frac{x-a}{b-a} f(a) + \frac{b-x}{b-a} f(b) \right] g(x)dx \\
(2.21) \quad &= \frac{f(a)+f(b)}{2} \int_a^b g(x)dx.
\end{aligned}$$

The inequalities (2.18) follow from (2.19), (2.20) and (2.21). ■

Remark 9. If we choose $g(x) \equiv 1$, then Theorem 11 reduces to Theorem 5.

Remark 10. If $s = 1$, then the inequality (2.17) reduces to the first inequality of (1.2) when $0 \leq a < b < \infty$.

Theorem 12. *Let f, a and b be defined as in Theorem 6. Then*

$$\begin{aligned}
 & 2^{s-1} f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \\
 & \leq \int_a^b f(x) g(x) dx \\
 (2.22) \quad & \leq \frac{f(a) + f(b)}{2} \int_a^b \left[\left(\frac{b-x}{b-a}\right)^s + \left(\frac{x-a}{b-a}\right)^s \right] g(x) dx.
 \end{aligned}$$

Proof. Since $f \in K_s^2$, g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned}
 & 2^{s-1} f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \\
 & = 2^{s-1} \int_a^b f\left(\frac{x}{2} + \frac{a+b-x}{2}\right) g(x) dx \\
 & \leq 2^{s-1} \int_a^b \left[\left(\frac{1}{2}\right)^s f(x) + \left(\frac{1}{2}\right)^s f(a+b-x) \right] g(x) dx \\
 & = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] g(x) dx \\
 & = \frac{1}{2} \left[\int_a^b f(x) g(x) dx + \int_a^b f(a+b-x) g(a+b-x) dx \right] \\
 (2.23) \quad & = \int_a^b f(x) g(x) dx.
 \end{aligned}$$

On the other hand, using (1.7) we have

$$\begin{aligned}
 & \frac{1}{2} \left[\int_a^b [f(x) + f(a+b-x)] g(x) dx \right] \\
 & = \frac{1}{2} \int_a^b \left[f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) + f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right) \right] g(x) dx \\
 & \leq \frac{1}{2} \int_a^b \left[\left(\frac{b-x}{b-a}\right)^s f(a) + \left(\frac{x-a}{b-a}\right)^s f(b) \right. \\
 & \quad \left. + \left(\frac{x-a}{b-a}\right)^s f(a) + \left(\frac{b-x}{b-a}\right)^s f(b) \right] g(x) dx \\
 (2.24) \quad & = \frac{f(a) + f(b)}{2} \int_a^b \left[\left(\frac{b-x}{b-a}\right)^s + \left(\frac{x-a}{b-a}\right)^s \right] g(x) dx.
 \end{aligned}$$

The inequalities (2.22) follow from (2.23) and (2.24). ■

Remark 11. *If we choose $g(x) \equiv 1$, then Theorem 12 reduces to Theorem 6.*

Remark 12. *If $s = 1$, then the inequalities (2.22) reduces the inequalities (1.2) when $0 \leq a < b < \infty$.*

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